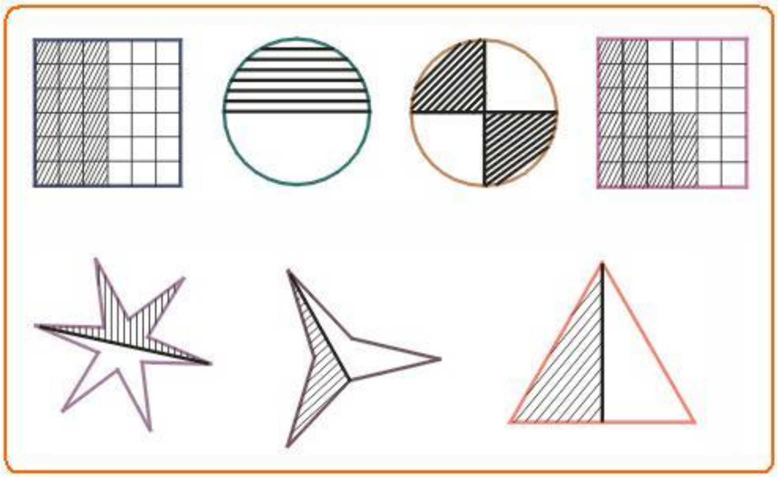
Congruent Triangles

Introduction

Geometrical figures having the same shape and size are said to be congruent. To verify whether two plane figures are congruent or not, place them one above the other (superimpose) and see if they match.

Study the following patterns:

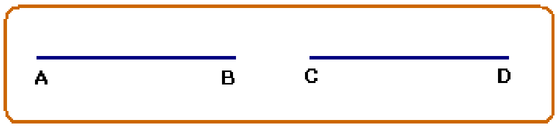


Drag shaded portion of each figure over the corresponding unshaded portion. If the shaded portion matches (coincides) with the corresponding unshaded portion, then that pair of figures are said to be congruent.

Recall the axiom on real numbers "if a and b are two real numbers, then one of the three relations, a = b or a > b or a**b or a**

Equality relation: a = b

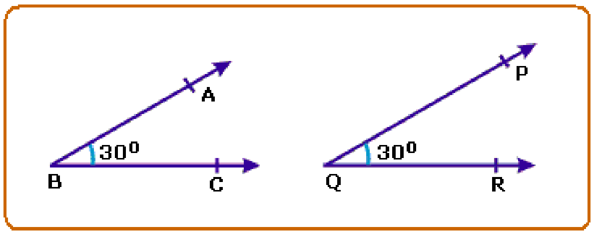
Two line segments of the same magnitude are congruent.



In the figure, segment AB = segment CD = 2cm.

We write this relation as $\overline{AB} \cong \overline{CD}$.

Angles of equal magnitude are congruent. We write this relation $A\hat{B}C \cong P\hat{Q}R$

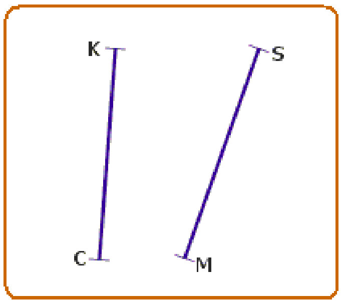


Relation a > b or a **< relations>**

Note:

There are many more inequality relations.

Congruent Line Segments

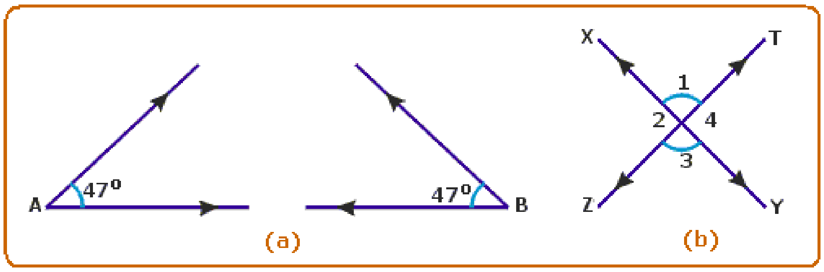


Drag $\overline{MS}$ over $\overline{CK}$. If $\overline{MS}$ coincides with CK then $\overline{MS}$ and $\overline{CK}$ are congruent line segments.

Line segments whose lengths are equal are congruent.

Conversely, congruent line segments have the same measure.

Congruent Angles



When two angles have the same measure, they are congruent.

In the above diagram (a), $\hat{A}$ and $\hat{B}$ are congruent angles since each angle is 47°.

Symbolically congruence relation is written as $\cong$.

$\hat{A} \cong \hat{B}$ means $\hat{A}$ is congruent to $\hat{B}$.

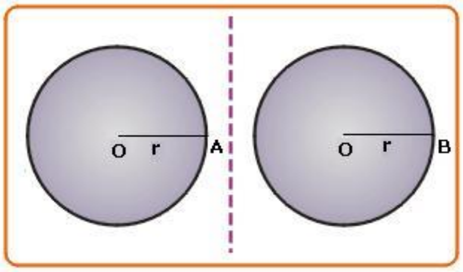
In the above diagram (b), vertically opposite angles are congruent.

Symbolically we can write the same as,

$\hat{1} \cong \hat{3}$ and $\hat{2} \cong \hat{4}$

Congruent Circles

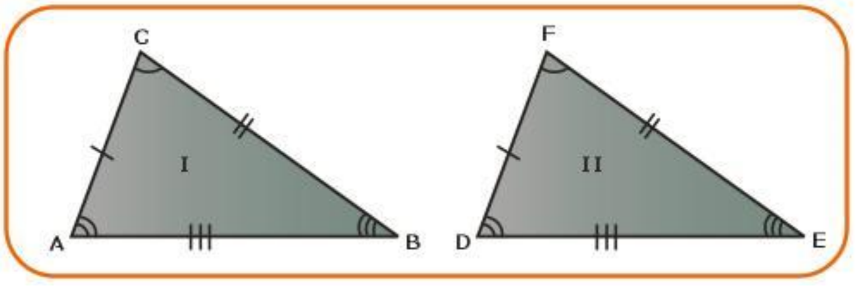
Two circles are said to be congruent if they have the same radius.



In the above figure, both the circles have the same radius OA = OB = r.

Congruent Triangles

Congruent triangles are triangles that have the same shape and size. The shape of a triangle is determined by the three angles of a triangle and the size of it is determined by the three sides.



Hence angles and sides determine the congruency of triangles.

Observe ΔABC and ΔDEF. Superimpose one over the other, they match each other. They are congruent triangles. In triangles ABC and DEF you will find that

$\hat{C}$ coincides with $\hat{F}$, $\overline{AC}$ coincides with $\overline{DF}$

$\hat{A}$ coincides with $\hat{D}$, $\overline{BC}$ coincides with $\overline{EF}$

$\hat{B}$ coincides with $\hat{E}$, $\overline{AB}$ coincides with $\overline{DE}$

Thus all six elements (3 angles and 3 sides) of one triangle should be congruent to the corresponding six elements of the other for the two triangles to be congruent.

Postulates of Congruency of Triangles

The above observation leads us to the following definition for congruency of two triangles.

Definition: "Two triangles are congruent if and only if all the sides and the angles of one are equal to the corresponding sides and angles of the other".

Sufficient Condition for Congruence of Two Triangles

From the definition of congruence of two triangles, we see that six conditions must be satisfied (3 sides and 3 angles) for two triangles to be congruent. But now, through activities we will find that if three conditions out of the six conditions are satisfied, the other three are automatically satisfied.

Activity

Given ABC and DEF are two triangles such that AB = DE = 4.5cm,

AC=DF=5.5cm and $B\hat{A}C = E\hat{D}F = 110^{0}$.

In these two triangles, we have two sides and their included angle of ΔABC equal to two corresponding sides and their included angle of ΔDEF.

Now drag ΔABC on ΔDEF. They will match each other.

$\therefore \Delta ABC \cong \Delta DEF$

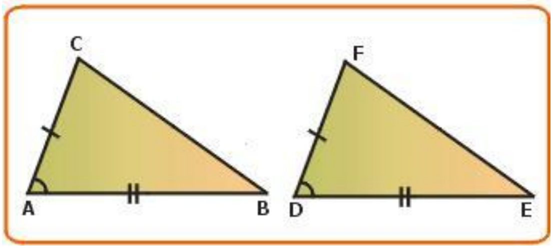
The result of this activity is stated as the SAS congruency postulate. SAS indicates Side-Angle-Side.

SAS Congruency Condition

"Two triangles are congruent if two sides and the included angle of one are equal to the corresponding sides and the included angle of the other"

Theorem 9:

"Two triangles are congruent if any two sides and the included angle of one triangle are equal to the corresponding sides and the included angle of the other triangle".



Given:

ABC and DEF are two triangles in which

AB = DE

AC = DF

$\hat{B} = \hat{E}$

To prove

$\Delta ABC \cong \Delta DEF$

Proof:

Place ΔABC on ΔDEF such that B falls on E, and BC along EF, since BC = EF, C falls on F.

Also $\hat{B} = \hat{E}$

∴ AB falls on DE.

∴ A coincides with D.

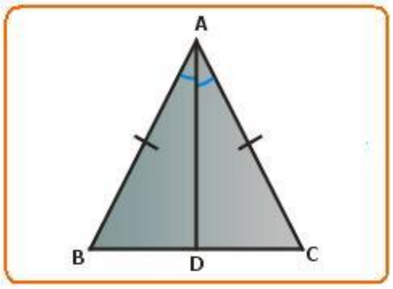
A coincides with D, C coincides with F.

∴ AC coincides with DF.

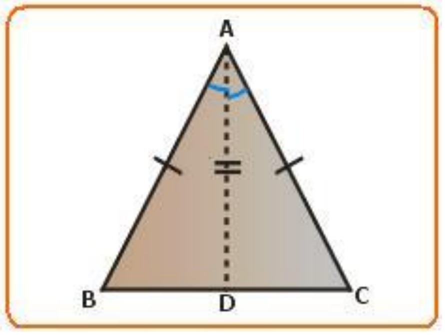
∴ ΔABC coincides with ΔDEF.

i.e., $\Delta ABC \cong \Delta DEF$

Application of SAS Congruency



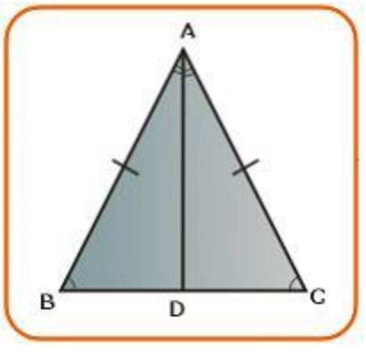
In order to establish a relation between $\hat{B}$ and $\hat{C}$, the given data is not enough. Therefore, some other additional details are required. For this we need to establish a logic which requires an analysis of the problem.



The SAS congruency condition establishes the relation between two triangles on the basis of angles and sides. Hence the additional details required for this figure is to draw the angle bisector of $B\hat{A}C$. Then we get two triangles which can be proved to be congruent. Therefore, the required construction is draw $\overline{AD}$, the bisector of $B\hat{A}C$.

Theorem 10:

"Angles opposite to equal sides are equal".



Given:

In ΔABC, AB = AC

To prove:

$A\hat{B}C = A\hat{C}B$

Construction

Draw AD, bisector of angle A.

Proof:

Compare triangles BAD and CAD.

AB=AC (Given)

$\overline{AD} = \overline{AD}$ (Common side)

$B\hat{A}D = C\hat{A}D$ (By construction)

$\therefore \Delta BAD \cong \Delta CAD$ (SAS congruency postulate)

$\therefore A\hat{B}D = A\hat{C}D$(c.p.c.t)

i.e., $\hat{B} = \hat{C}$

Hence the theorem is proved.

ASA congruence condition

Activity

Construct triangle ABC such that AB=5cm, $\hat{A} = 30^0$ and $\hat{B} = 50^0$.

Construct another Δ DEF such that DE=5cm, $\hat{D} = 30^0$ and $\hat{E} = 50^0$.

Drag ΔABC on ΔDEF.

We will notice that ΔABC matches with ΔDEF. This means

$\Delta ABC \cong \Delta DEF$ if $\overline{AB} = \overline{DE}$, $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$.

Hence the Conclusion

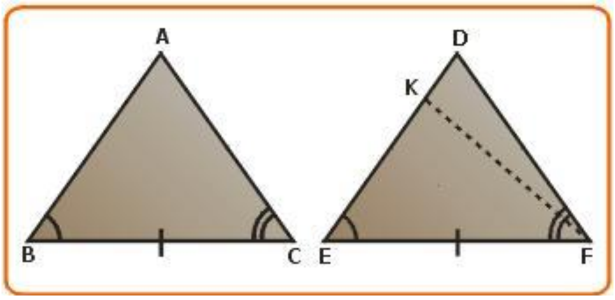
"Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle".

This is one of the congruence conditions and is called the ASA congruence condition. ASA indicates Angle-Side-Angle. This condition establishes congruence of two triangles having corresponding two angles and included side congruent.

Let us prove this.

Theorem 11:

"Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle".



Given:

In triangles ABC and DEF,

$\angle B = \angle E$

$\angle C = \angle F$

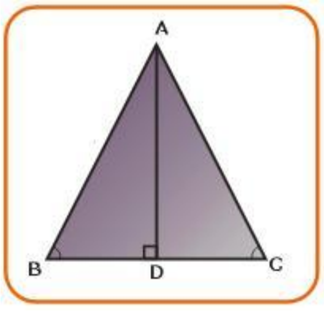
BC=EF

To prove

$\Delta ABC \cong \Delta DEF$

Proof:
There are three possibilities
Case I: AB = DE
Case II: AB < DE
Case III: AB > DE
Case I: In addition to data, if AB = DE then
$\Delta ABC \cong \Delta DEF$
(by SAS congruence postulate)
Case II: If AB < such DE on K point take then>
that EK = AB. Join KF.
Now compare triangles ABC and KCF,
BC = EF (given)
$\hat{B} = \hat{E}$(given)
AB = EK (supposed)
$\therefore$ ABC $\cong \Delta$KEF (SAS criterion)
Hence $A\hat{C}B = K\hat{F}E$
But $A\hat{C}B = D\hat{F}E$ (given)
$K\hat{F}E = D\hat{F}E$.
This can happen only if K coincides with D.
$\therefore$ AB must be equal to DE.

Case III: If AB > DE, then a similar argument applies.
AB must be equal to DE.
Hence the only possibility is that AB must be equal to DE and from SAS congruence condition $\Delta ABC \cong \Delta DEF$.
Hence the theorem is proved.

Theorem 12:
"In a triangle the sides opposite to equal angles are equal".
This theorem can also be stated as
"The sides opposite to equal angles of a triangle are equal".


Given

In ΔABC, $\hat{B} = \hat{C}$

To prove:

$\overline{AB} = \overline{AC}$

Construction:

Draw $\overline{AD} \perp \overline{BC}$

Proof:

By construction $A\hat{D}B = A\hat{D}C = 90^0$ that is triangles ADB and ADC are two right-angled triangles.

In triangles ADB and ADC,

$\hat{B} = \hat{C}$ (Given)

$A\hat{D}B = A\hat{D}C = 90^0$ (by construction)

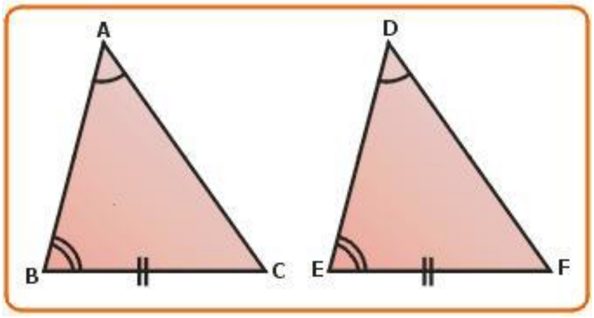
$\overline{AD}$ is common to both the triangles.

$\therefore \Delta ADB \cong \Delta ADC$ (A.S.A. postulate)

$\therefore \overline{AB} = \overline{AC}$ (corresponding sides)

AAS Congruence Condition

If instead of two angles and an included side, two angles and a non-included side of one triangle is equal to the corresponding angles and side of another triangle the two triangles be congruent.



Given:

In triangles ABC and DEF,

BC = EF (non-included sides)

$\hat{B} = \hat{E}$

$\hat{A} = \hat{D}$

To prove

$\Delta ABC \cong \Delta DEF$

Proof:

$\hat{B} = \hat{E}$ (Given)

$\hat{A} = \hat{D}$ (Given)

$\hat{A} + \hat{B} = \hat{D} + \hat{E}$ ....(i)

Since $\hat{A} + \hat{B} + \hat{C} = \hat{E} + \hat{D} + \hat{F} = 180^0$ and $\hat{A} + \hat{B} = \hat{D} + \hat{E}$

(from (i))

$\therefore \hat{C} = \hat{F}$ ....(ii)

Now in triangles ABC and DEF,

$\hat{B} = \hat{E}$ (given)

$\hat{C} = \hat{F}$ (Proved from (ii))

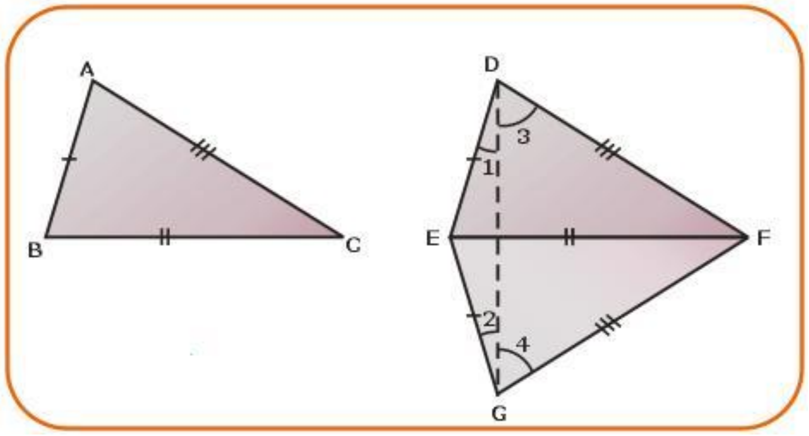
BC = EF (given)

$\therefore \Delta ABC \cong \Delta DEF$ (SAS congruence)

SSS Congruence Condition

Statement:

"Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle".



Given:

In triangles ABC and DEF,

AB = DE

BC = EF

AC = DF

Note:

Let BC and EF be the longest sides of

ΔABC, ΔDEF respectively.

To prove:

$\Delta ABC \cong \Delta DEF$

Construction: If BC is the longest side, draw EG such that $EG = AB$ and $G\hat{E}F = A\hat{B}C$.
Join GF and DG.

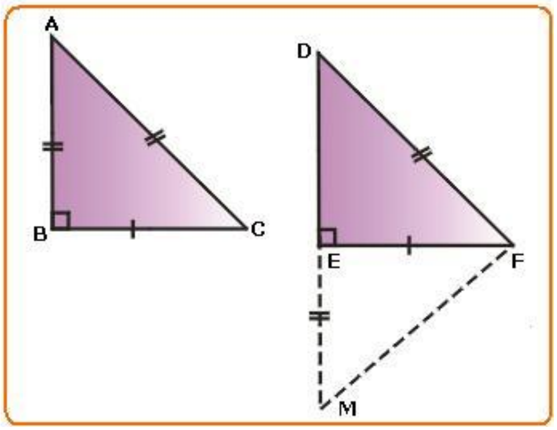
Proof:

In triangles ABC and GEF,
AB = GE (by construction)
BC = EF (given)
$A\hat{B}C = G\hat{E}F$ (by construction)
$\therefore \Delta ABC \cong \Delta GEF$
(SAS congruence condition)
$\therefore \quad B\hat{A}C = E\hat{G}F$ (CPCT)
and AC = GF (CPCT)
AB = GE (by construction)
But AB = DE (given)
$\therefore$ DE = GE
Similarly, DF = GF
In Δ EDG,
DE = GE (Proved)
$\therefore \angle 1 = \angle 2 \quad \ldots (i)$
(angles opposite equal sides)
In Δ DFG,
DF = GF (Proved)
$\therefore \angle 3 = \angle 4 \quad \ldots (ii)$
(angles opposite equal sides)
By adding (i) and (ii), we get
$\angle 1 + \angle 3 = \angle 2 + \angle 4$
i.e., $E\hat{D}F = E\hat{G}F$
But $E\hat{G}F = B\hat{A}C$ (Proved)
$\therefore E\hat{D}F = B\hat{A}C$.
Now in triangles ABC and DEF,
AB = DE (given)
AC = DF (given)
$B\hat{A}C = E\hat{D}F$ (Proved)
$\therefore \Delta ABC \cong \Delta DEF$
(SAS congruency condition)

RHS (Right Angle Hypotenuse Side) Congruence Theorem

Statement:

Two right-angled triangles are congruent if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and the corresponding side of the other triangle.



Given:

ABC and DEF are two right-angled triangles such that

i) $\angle B = \angle E = 90^0$

ii) Hypotenuse $\overline{AC}$ = Hypotenuse $\overline{DF}$ and

iii) Side BC = side EF

To prove:

$\Delta ABC \cong \Delta DEF$

Construction

Produce DE to M so that EM = AB. Join MF.

Proof:

In triangles ABC and MEF,

AB = ME (construction)

BC = EF (given)

$A\hat{B}C = M\hat{E}F$ each 90^0

$\therefore \Delta ABC \cong \Delta MEF$

(SAS congruency condition)

Hence $\hat{A} = \hat{M}$ ...(i) (c.p.c.t)

$\overline{AC} = \overline{MF}$ ...(ii) (c.p.c.t)

Also AC = DF …(iii) (given)

From (i) and (iii),

$\therefore$ DF = MF

$\therefore \hat{D} = \hat{M}$…(iv)

(Angles opposite to equal sides of Δ DFM)

From (ii) and (iv),

$\hat{A} = \hat{D}$ ...(v)

Now compare triangles ABC and DEF,

$\hat{A} = \hat{D}$ (from (vi))

$\hat{B} = \hat{E}$ (given)

$\therefore \hat{C} = \hat{F}$ ...(vii)

Compare triangles ABC and DEF,

$\overline{BC} = \overline{EF}$ (given)

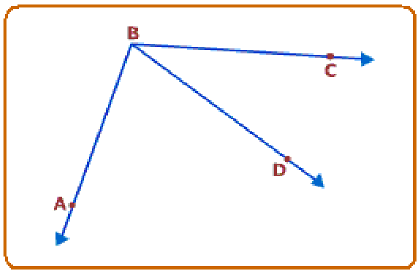
$\overline{AC} = \overline{DF}$ (given)

$\hat{C} = \hat{F}$ (from (vii))

$\therefore \Delta ABC \cong \Delta DEF$

(SAS congruence condition)

Inequality of Angles



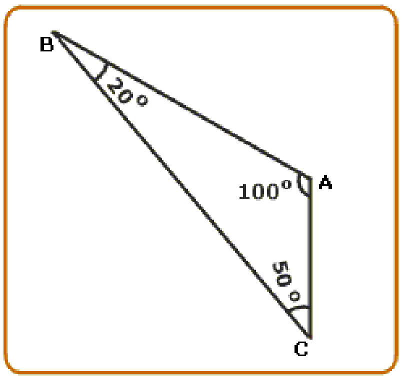
In figure, observe that $A\hat{B}C > D\hat{B}C$ (Whole is greater than its parts).

Also, $A\hat{B}C > A\hat{B}D$.

We may also write these inequality relations as

$D\hat{B}C < A\hat{B}C : A\hat{B}D < A\hat{B}C$.

Inequality in a Triangle



Activity

Construct a triangle ABC as shown in the figure.

Observe that in Δ ABC,

$\overline{AC}$ is the smallest side (2cm)

B is the angle opposite to $\overline{AC}$ and $\hat{B} = 20^0$

$\overline{BC}$ is the greatest side (6cm)

A is the angle opposite to BC and $\hat{A} = 100^0$

From the measurements made above of side and angle opposite to it, we can write the relation in the form of a statement.

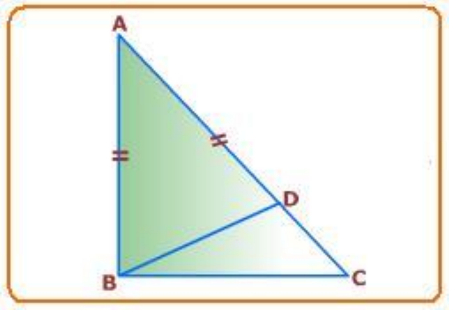
"If two sides of a triangle are unequal then the longer side has the greater angle opposite to it".

Theorem on Inequalities

Theorem 1:

Statement:

If two sides of a triangle are unequal, the longer side has the greater angle opposite to it.



Read the statement and draw a triangle as per data.

Draw ΔABC, such that AC > AB.

Data:

In ΔABC, $\overline{AC} > \overline{AB}$

To Prove:

$A\hat{B}C > A\hat{C}B$

Construction:

Take a point D on $\overline{AC}$ such that $\overline{AB} = \overline{AD}$. Join B to D.

Proof:

In ΔABD, $\overline{AB} = \overline{AD}$ (by construction)

$\therefore \quad A\hat{B}D = A\hat{D}B \qquad \ldots (i)$

but, $A\hat{D}B$ is the exterior angle with reference to ΔDBC. Assume $\overline{CD}$ is produced

$\therefore \quad A\hat{D}B > D\hat{C}B \qquad \ldots (ii)$

From relations (i) and (ii) we can write $A\hat{B}D > D\hat{C}B$.

But $A\hat{B}D$ is only a part of $A\hat{B}C$.

$\therefore$ $A\hat{B}C > D\hat{C}B$ or $A\hat{B}C > A\hat{C}B$

Hence the theorem is proved.

Analysis for Construction

From the given data and what is to be proved, we come to know that the relation between exterior angle and interior angles of a triangle can be used to prove this theorem. Hence the required construction would

be to get an exterior angle within the Δ ABC by taking a point D on $\overline{AC}$,

such that $\overline{AD} = \overline{AB}$.

Angle Side Relation

Theorem 2:

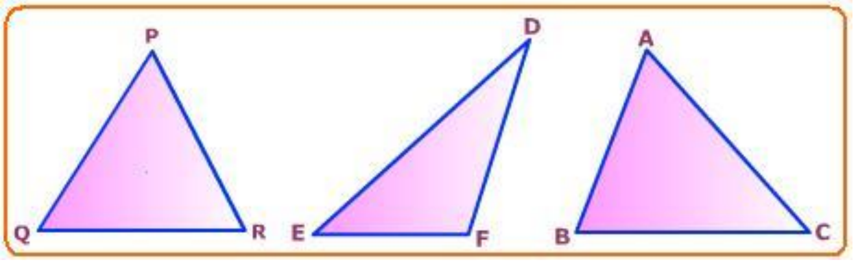
Statement:

In a triangle, if two angles are unequal, the side opposite to greater angle is longer than the side opposite to the smaller angle.

An activity is suggested to understand this theorem.

Activity

Draw three triangles with measures given in figure. Measure the sides and tabulate the measurement as shown in the table.



Triangle	ΔPQR			ΔDEF			ΔABC		
Angles	P	Q	R	D	E	F	A	B	C
Angles Measure	60^0	40^0	80^0	40^0	30^0	110^0	65^0	75^0	40^0
Opposite Sides	QR	PR	PQ	EF	DF	DE	BC	AC	AB
Side measure	3.5 cm	3 cm	4.4 cm	4.5 cm	3.8cm	7 cm	4 cm	4.4 cm	2.9 cm

In ΔPQR, $\hat{R} > \hat{Q}$, sides opposite to $\hat{R}$ and $\hat{Q}$ are PQ and PR respectively.

Here the angle side relation is PQ > PR.

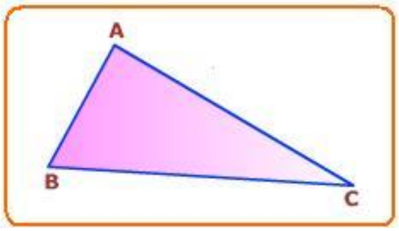
Similarly $\hat{R} > \hat{P}$, PQ > QR

Similarly, write the angle-side relation with reference to ΔDEF and ΔABC from the table of measurement, we can write the angle side relation as follows:
"In a triangle, if two angles are unequal, the side opposite to the greater angle is longer than the side opposite to the smaller angle".

Theorem 3

Statement:

In a triangle, the greater angle has the longer side opposite to it.



Given:

In ΔABC, $A\hat{B}C > A\hat{C}B$

To prove:

AC > AB

Proof:

In ΔABC, AB and AC are two line segments. So the following are the three possibilities of which exactly one must be true.
(i) either AB = AC or
(ii) AB > AC
(iii) AB < AC < td>

(i) Now if AB=AC, then $\angle B = \angle C$ which is contrary to the hypothesis.

$\therefore$ AB $\neq$ AC

(ii) If AB>AC, then $\angle C > \angle B$ which is also contrary to the hypothesis.

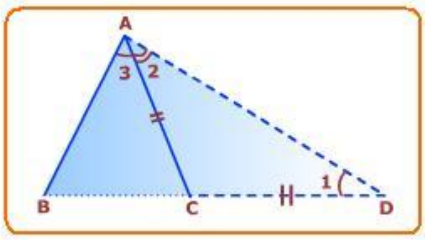
$\therefore$ AB $\not>$ AC

(iii) We are therefore left with only one possibility namely AB < AC.< td>
$\therefore$ AC > AB must be true.
$\therefore$ AC > AB
Hence the theorem is proved.

Theorem 4

Statement:

Prove that in any triangle the sum of the lengths of any two sides of a triangle is greater than the length of its third side.
Draw ΔABC.



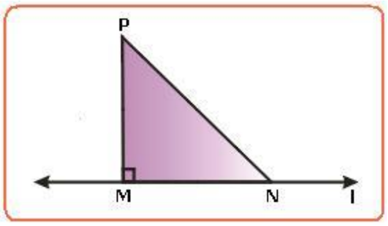
Data:
ABC is a triangle.
To prove:
$\overline{AB} + \overline{AC} > \overline{BC}$
$\overline{AB} + \overline{BC} > \overline{AC}$
$\overline{AC} + \overline{BC} > \overline{AB}$
Construction:
Produce $\overline{BC}$ to D such that $\overline{AC} = \overline{CD}$. Join A to D.
Proof:
AC = CD (by construction)

$\therefore\ \hat{1} = \hat{2}$ …(i)
From the figure,

$B\hat{A}D = \hat{2} + \hat{3}$ …(ii)
But $B\hat{A}D > \hat{2}$
$\therefore B\hat{A}D > \hat{1}$ (Proved from (i))

Side opposite to $\hat{1}$ is $\overline{AB}$ and side opposite to $B\hat{A}D$ is $\overline{BD}$.
$\therefore$ In ΔABD, $\hat{2} + \hat{3} > \hat{1}$
BD > AB (side opposite to greater angle is greater)
From figure BD = BC + CD
$\therefore\ BC + CD > AB$
$BC + AC > AB$
$\because$CD = AC (by construction)
Hence sum of two sides of a triangle is greater than the third side.
Similarly we can prove $AB + AC > BC$ and $AB + BC > AC$
Hence the theorem is proved.

Theorem 5
Statement:
Of all the line segments that can be drawn to a given line, from a point not lying on it, the perpendicular line segment is the shortest.



Given:

l is a line and P is a point not lying on l. PM ⊥ l. N is any point on l other than M.

To prove:

PM < PN

Proof:

In Δ PMN, ∠M is the right angle.

∴ ∠N is an acute angle (Angle sum property of triangle)

∵ ∠M > ∠N

PN > PM (Side opposite greater angle)

or PM < PN